Solution Set 9, 18.06 Fall '12

1. Do Problem 7 from 6.4.

   Solution. (a) The determinant is $1 - b^2$. If $b = 2$ this is negative which implies that the two eigenvalues are of opposite signs (determinant=product of eigenvalues), in particular one of them must be negative.

   (b) The number of negative pivots is the same as the number of negative eigenvalues.

   (c) This matrix has trace 2 whatever b is so it cannot have two negative eigenvalues (trace=sum of eigenvalues). □

2. Do Problem 26 from 6.4.

   Solution. We see that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $1 + 10^{-15}$. The angle between the two eigenvectors is 45 degrees. □

3. Do Problem 4 from 6.5.

   Solution. For the first one, we find :

   $$f = x^2 + 9y^2 + 4xy$$

   This can be written as a sum of two squares :

   $$f = (x + 2y)^2 - 4y^2 + 9y^2 = (x + 2y)^2 + 5y^2$$

   For the second one, we find :

   $$f = x^2 + 9y^2 + 6xy$$

   This can be written as one square :

   $$f = (x + 3y)^2$$

   □

4. Do Problem 16 from 6.5.

   Solution. Recall that a positive definite matrix is defined to be a matrix A such that $\mathbf{x}^T A\mathbf{x} > 0$ for any non zero vector $\mathbf{x}$.

   We see that for the matrix A given in this problem, when $x_1 = x_3 = 0$ and $x_2 = 1$, $\mathbf{x}^T A\mathbf{x} = 0$, therefore A fails to be a definite positive matrix. □

5. Do Problem 30 from 6.5.

Solution. The expression $z = ax^2 + 2bxy + cy^2$ can be identified with $\mathbf{x}^T A \mathbf{x}$ where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. A saddle point will happen when this expression can take both positive and negative values. This happens if A is neither positive definite nor negative definite. Equivalently this happens when A has one negative eigenvalue and one positive eigenvalue. Since the determinant is the product of the eigenvalue, a simple way to test this fact is to check that the determinant is negative.

In conclusion, the function z has a saddle point at $(0, 0)$ if $ac - b^2 < 0$. □

6. Do Problem 2 from 6.6

Solution. Let $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. M is a permutation matrix, whose inverse is M itself. Multiplying a 2-by-2 matrix on the left by M swaps the two rows and multiplying on the right swaps the two columns. In particular, we see that :

$$M \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} M^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

which shows that the two matrices are similar. □

7. Do Problem 12 from 6.6.

Solution. Let M be any 4x4 matrix. Let's call its rows R_1, R_2, R_3, R_4 and its columns C_1, C_2, C_3, C_4. With this notation, $JM = \begin{bmatrix} R_2 \\ 0 \\ R_4 \\ 0 \end{bmatrix}$ and $MK = \begin{bmatrix} 0 & C_1 & C_2 & 0 \end{bmatrix}$. Saying that these two matrices are equal tells us that some entries of M have to be 0. More precisely, the first and second column have a zero in second and fourth position and the second and fourth row have a zero in the first and fourth position. M has to be of the following form :

$$M = \begin{bmatrix} ? & ? & ? & ? \\ 0 & 0 & ? & 0 \\ ? & ? & ? & ? \\ 0 & 0 & ? & 0 \end{bmatrix}$$

We see that the second and fourth row of M are dependant which prevents M from being invertible.

Remark : Another way to see that J and K are not similar is to compute their square $J^2 = 0$ and $K^2 \neq 0$. □

8. Do Problem 8 from 6.7.

Solution. The matrix Σ is diagonal with positive entries on the diagonal. The matrix Σ^{-1} is obtained by taking the inverse of each diagonal entry. Taking inverse reverses inequality, therefore $\sigma_{\max}(A^{-1})$ which is the biggest number among the inverses of the singular values of A must be the inverse of the smallest singular value of A :

$$\sigma_{\max}(A^{-1}) = \sigma_{\min}(A)^{-1}$$

Thus the product $\sigma_{\max}(A^{-1})\sigma_{\max}(A)$ is equal to the quotient $\frac{\sigma_{\max}(A)}{\sigma_{\min}(A)}$. Since $\sigma_{\max}(A) \geq \sigma_{\min}(A) > 0$, we must have :

$$\frac{\sigma_{\max}(A)}{\sigma_{\min}(A)} \geq 1$$

□

9. Do Problem 13 from 6.7.

Solution. Let $R = USV^T$ be the SVD of R, and $A = U'S'V'^T$ be the SVD of $A = QR$. We know that the diagonal entries of S are the square roots of the eigenvalues of R^TR and the columns of V are an orthonormal basis of eigenvectors of R^TR. We have :

$$A^TA = R^TQ^TQR = R^TR$$

This implies that $S' = S$ and $V = V'$.

We see that if we take $U' = QU$ U' is orthogonal and we have :

$$A = QUSV^T$$

which shows that $U' = QU$.

In conclusion, only U is changed because of Q. □

10. The solution is on the MATLAB solutions file.